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Complementarity relation between the USp(2ν) and SO*($2d$) Lie groups

C Quesne†

Physique Théorique et Mathématique CP 229, Université Libre de Bruxelles, Boulevard du Triomphe, B 1050 Brussels, Belgium

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Abstract. The object of this paper consists in finding a complementary group with respect to USp(n) within either irreducible representation $\langle(1/2)^{dn}\rangle$ or $\langle(1/2)^{dn-1}3/2\rangle$ of the real symplectic group Sp($2dn, R$). For such purpose, we deal with the (full) orthogonal group O(n), $n=2\nu$ or $2\nu+1$, and the unitary symplectic group USp(n), $n=2\nu$, in a unified way by introducing an appropriate metric. A similar treatment applied to their complementary group determination leads to the well known Sp($2d, R$) group in the O(n) case, and to the SO*($2d$) group in the USp(n) case.

1. Introduction

Since Schwinger's (1952) pioneering work on SU(2), it has become customary to realise the generators of Lie groups in terms of boson operators. Such a realisation enabled Moshinsky (1963) and Baird and Biedenharn (1963) to construct bases for the irreducible representations (irreps) of the unitary group U(n). In his derivation, Moshinsky used a special type of relation between the unitary groups U(n) and U(d) within the symmetrical irreps of U(dn), which was generalised to arbitrary Lie groups and termed complementarity by Moshinsky and Quesne (1970). Later on, such a relation was dealt with in Howe's duality theory (1979), where it was referred to as a duality correspondence (Gelbart 1979).

The object of this paper consists in finding a complementary group with respect to the unitary symplectic group USp(n) ($n=2\nu$ even), when the latter is embedded into the real symplectic group Sp($2dn, R$). The result of this development matches those obtained from another approach by Gross and Kunze (1977) and Gelbart (1979).

In § 2, we recall the definition and main properties of complementary groups, and review the various examples known in the physical literature. In § 3, we construct the generators of a direct product subgroup $G_1\times G_2$ of Sp($2dn, R$), where G_1 is USp(n), and G_2 is shown to be SO*($2d$). Finally, in § 4, we establish the complementarity of USp(n)and SO*($2d$).

2. Complementary groups

Two groups G_1 and G_2, whose direct product is contained in a larger group H, are referred to as complementary within a definite irrep μ of H if there is a one-to-one

† Maître de recherches FNRS.

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correspondence between all the irreps λ_1 and λ_2 of G_1 and G_2 contained in this irrep of H (Moshinsky and Quesne 1970). Note that the complementary relation is symmetrical; hence, in the following, G_1 and G_2 may be interchanged at will.

Such a definition leads to the following important consequences:

(i) In the decomposition of μ into irreps of G_1, a given irrep λ_1 of the latter appears with a multiplicity equal to the dimension of the corresponding irrep λ_2 of the complementary group G_2.

(ii) In the carrier space of μ, the set of vectors $\varphi^{\mu}_{\lambda_1 r_1 \alpha}$ transforming under G_1 according to equivalent irreps λ_1, and belonging to a given row r_1 of the latter (e.g. the set of highest weight states of the equivalent irreps λ_1) span the carrier space of the G_2 corresponding irrep λ_2; the multiplicity index α may therefore be chosen as $\lambda_2 r_2$, where λ_2 is entirely determined by λ_1 (hence redundant), and r_2 denotes the row of λ_2.

(iii) When restricted to the carrier space of μ, the Casimir operators of G_2 functionally depend upon those of G_1.

In the physical literature, various pairs of complementary groups are known among the classical Lie groups, when their generators are realised in terms of boson operators. As mentioned in the introduction, the U(n) and U(d) subgroups of U(dn) form such a pair.

For the (full) orthogonal group O(n), Chacón (1969) and Moshinsky and Quesne (1971) proved the existence of a complementary group, namely the subgroup Sp($2d$, R) of the real symplectic group Sp($2dn$, R). As discussed in a recent paper (Couvreur *et al* 1983), this complementarity relation does not hold true in general when O(n) is replaced by its rotation subgroup SO(n).

For the unitary symplectic group USp(n) (n even), Quesne (1973) proved the existence of a complementary group, which is again a non-compact subgroup of Sp($2dn$, R). However, its precise nature was not determined beyond the fact that its root diagram was shown to belong to Cartan's D_l class (with $l = d$). The aim of the present paper is to fill in this gap. By introducing an appropriate metric, we shall treat both O(n) and USp(n), and their corresponding complementary groups in a unified way. Thence, as it will be shown elsewhere, all known results in the O(n) representation theory (see e.g. Deenen and Quesne 1983, Quesne 1984a, b) can be easily extended to USp(n).

3. Direct product group USp(2ν)×SO*($2d$)

Let us take for G_1 either the orthogonal group O(n) or the unitary symplectic group USp(n). They are made of those $n\times n$ matrices $\boldsymbol{A}$ which respectively preserve a bilinear symmetrical or antisymmetrical metric $\boldsymbol{g}$, i.e., satisfy the condition $\boldsymbol{A}\boldsymbol{g}\tilde{\boldsymbol{A}} = \boldsymbol{g}$, where ˜ stands for transposed. We can accommodate both choices for $\boldsymbol{g}$ by choosing the symmetry condition $\tilde{\boldsymbol{g}} = \varepsilon \boldsymbol{g}$, where $\varepsilon = \pm 1$. In addition, we assume $\boldsymbol{g}$ to be normalised according to the relation $\boldsymbol{g}\tilde{\boldsymbol{g}} = \boldsymbol{I}$. The O($n$) and USp($n$) groups will be designated by the common notation G(n). For the orthogonal group, n may be either even or odd ($n = 2\nu$ or $2\nu + 1$), whereas for the symplectic one, it must be even ($n = 2\nu$).

The G(n) generators Λ_{st}, $s, t = 1, \ldots, n$†, can be realised in terms of dn boson creation operators η_{is}, $i = 1, \ldots, d$, $s = 1, \ldots, n$, and the corresponding annihilation

† For $\varepsilon = +1$, we actually deal with the Lie algebra of SO(n).

operators ξ_{is}, as follows (Lohe 1974):

$$\Lambda_{st} = \sum_{p=1}^{n} (g_{sp}E_{pt} - \varepsilon g_{tp}E_{ps}), \tag{1}$$

where the operators

$$E_{st} = \frac{1}{2} \sum_{i=1}^{d} (\eta_{is}\xi_{it} + \xi_{it}\eta_{is}) \tag{2}$$

are the generators of the U(n) group whereof G(n) is a subgroup. The $\Lambda_{\sigma t}$ generator commutation relations can be written as

$$[\Lambda_{st}, \Lambda_{s't'}] = g_{s's}\Lambda_{t't} + g_{t't}\Lambda_{s's} + g_{s't}\Lambda_{st'} + g_{t's}\Lambda_{ts'}, \tag{3}$$

while their symmetry and Hermiticity properties are respectively given by

$$\Lambda_{st} = -\varepsilon\Lambda_{ts}, \qquad \text{and} \qquad (\Lambda_{st})^{\dagger} = \sum_{pq} g_{tp}\Lambda_{pq}g_{qs}. \tag{4}$$

G(n) can be embedded into the real symplectic group Sp($2dn$, R), generated by the bilinear operators (Moshinsky and Quesne 1971)

$$D^{\dagger}_{is,jt} = \eta_{is}\eta_{jt}, \qquad D_{is,jt} = \xi_{is}\xi_{jt}, \tag{5}$$

$$E_{is,jt} = \tfrac{1}{2}(\eta_{is}\xi_{jt} + \xi_{jt}\eta_{is}). \tag{6}$$

Since the operators E_{st} of (2) are obtained by contracting the U(dn) generators of (6) with respect to index i, we obtain the group chain

$$\mathrm{Sp}(2dn, R) \supset \mathrm{U}(dn) \supset \mathrm{U}(n) \supset \mathrm{G}(n). \tag{7}$$

Starting from the well known Sp($2dn$, R) commutation relations

$$\begin{aligned}
&[E_{is,jt}, E_{i's',j't'}] = \delta_{ji'}\delta_{ts'}E_{is,j't'} - \delta_{ij'}\delta_{st'}E_{i's',jt},\\
&[E_{is,jt}, D^{\dagger}_{i's',j't'}] = \delta_{ji'}\delta_{ts'}D^{\dagger}_{is,j't'} + \delta_{jj'}\delta_{tt'}D^{\dagger}_{i's',is},\\
&[E_{is,jt}, D_{i's',j't'}] = -\delta_{ii'}\delta_{ss'}D_{jt,j't'} - \delta_{ij'}\delta_{st'}D_{i's',jt},\\
&[D^{\dagger}_{is,jt}, D^{\dagger}_{i's',j't'}] = [D_{is,jt}, D_{i's',j't'}] = 0,\\
&[D_{is,jt}, D^{\dagger}_{i's',j't'}] = \delta_{ii'}\delta_{ss'}E_{j't',jt} + \delta_{ij'}\delta_{st'}E_{i's',jt} + \delta_{ji'}\delta_{ts'}E_{j't',is} + \delta_{jj'}\delta_{tt'}E_{i's',is},
\end{aligned} \tag{8}$$

we may now look for those linear combinations of the Sp($2dn$, R) generators which commute with the G(n) generators Λ_{st}. By contracting the Sp($2dn$, R) generators with respect to index s, and using the g metric, we simply obtain

$$D^{\dagger}_{ij} = \sum_{st} g_{st}D^{\dagger}_{is,jt}, \qquad D_{ij} = \sum_{st} g_{st}D_{is,jt}, \qquad E_{ij} = \sum_{s} E_{is,js}. \tag{9}$$

It is straightforward to check that

$$[\Lambda_{st}, D^{\dagger}_{ij}] = [\Lambda_{st}, D_{ij}] = [\Lambda_{st}, E_{ij}] = 0. \tag{10}$$

The set of operators $D^{\dagger}_{ij}$, D_{ij}, and E_{ij} closes under commutation as follows:

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}, \qquad [D^{\dagger}_{ij}, D^{\dagger}_{kl}] = [D_{ij}, D_{kl}] = 0, \tag{11a}$$

$$[E_{ij}, D^{\dagger}_{kl}] = \delta_{jk}D^{\dagger}_{il} + \delta_{jl}D^{\dagger}_{ki}, \qquad [E_{ij}, D_{kl}] = -\delta_{ik}D_{jl} - \delta_{il}D_{kj}, \tag{11b}$$

$$[D_{ij}, D^{\dagger}_{kl}] = \delta_{ik}E_{lj} + \varepsilon\delta_{il}E_{kj} + \varepsilon\delta_{jk}E_{li} + \delta_{jl}E_{ki}. \tag{11c}$$

Moreover, its symmetry and Hermiticity properties are respectively given by

$$D^{\dagger}_{ij} = \varepsilon D^{\dagger}_{ji}, \qquad D_{ij} = \varepsilon D_{ji}, \tag{12}$$

and

$$(E_{ij})^{\dagger} = E_{ji}, \qquad (D^{\dagger}_{ij})^{\dagger} = D_{ij}. \tag{13}$$

We shall denote by $G^c(2d)$ the Lie group generated by $D^{\dagger}_{ij}$, D_{ij}, and E_{ij}; subsequently $G^c(2d)$ will play the role of the G_2 group. In addition to the group chain (7), we also obtain the two chains

$$\mathrm{Sp}(2dn, R) \supset G^c(2d) \supset U(d), \tag{14}$$

and

$$\mathrm{Sp}(2dn, R) \supset G(n) \times G^c(2d), \tag{15}$$

where $U(d)$ is generated by the E_{ij} operators.

For $O(n)$, i.e. $\varepsilon = +1$, (11), (12), and (13) define the well known Lie algebra of $\mathrm{Sp}(2d, R)$. For $\mathrm{USp}(n)$, i.e. $\varepsilon = -1$, we shall proceed to prove that those equations characterise the Lie algebra of the rather unusual non-compact group $SO^*(2d)$ (Gilmore 1974, Günaydin and Saçlioğlu 1982, Klimyk and Gavrilik 1984).

For such a purpose, let us start from the compact $SO(2d)$ group generators $L_{\rho\sigma} = -L_{\sigma\rho} = -(L_{\rho\sigma})^{\dagger}$, $\rho, \sigma = 1, \ldots, 2d$, satisfying commutation relations similar to (3) (with $\boldsymbol{g} = \boldsymbol{I}$), and replace the ρ index by a pair of indices $i\alpha$, taking the values $i = 1, \ldots, d$, and $\alpha = 1, 2$. The following linear combinations of $L_{i\alpha, j\beta}$,

$$\begin{aligned} \bar{D}^{\dagger}_{ij} &= \tfrac{1}{2}[-iL_{i1,j1} + L_{i1,j2} + L_{i2,j1} + iL_{i2,j2}], \\ \bar{D}_{ij} &= \tfrac{1}{2}[-iL_{i1,j1} - L_{i1,j2} - L_{i2,j1} + iL_{i2,j2}], \\ \bar{E}_{ij} &= \tfrac{1}{2}[L_{i1,j1} - iL_{i1,j2} + iL_{i2,j1} + L_{i2,j2}], \end{aligned} \tag{16}$$

satisfy (11), (12), and (13) where $\varepsilon = -1$, except for (11c) which is replaced by

$$[\bar{D}_{ij}, \bar{D}^{\dagger}_{kl}] = -\delta_{ik}\bar{E}_{lj} + \delta_{il}\bar{E}_{kj} + \delta_{jk}\bar{E}_{li} - \delta_{jl}\bar{E}_{ki}. \tag{11c'}$$

The operators $\bar{E}_{ij}$ generate the $U(d)$ subgroup of $SO(2d)$.

The $SO^*(2d)$ Lie algebra is now obtained from that of $SO(2d)$ by applying the Weyl unitary trick to the generators $\bar{D}^{\dagger}_{ij}$ and $\bar{D}_{ij}$, belonging to the subspace orthogonal to the algebra of $U(d)$ (Gilmore 1974). In this process, (11c') is converted into (11c), while the other equations defining the Lie algebra remain unchanged, which completes the proof.

Note that for some low d values, $SO^*(2d)$ is locally isomorphic to other Lie groups. For $d = 2$, we indeed have $SO^*(4) \simeq SU(2) \times SU(1,1)$, where $SU(2)$ and $SU(1,1)$ are respectively generated by

$$J_+ = E_{12}, \qquad J_- = E_{21}, \qquad J_0 = \tfrac{1}{2}(E_{11} - E_{22}),$$

and (17)

$$K_+ = D^{\dagger}_{12}, \qquad K_- = D_{12}, \qquad K_0 = \tfrac{1}{2}(E_{11} + E_{22}),$$

and similarly for $d = 3$, $SO^*(6) \simeq SU(3,1)$.

4. Complementarity of the USp(2ν), SO*(2d) pair

Let us consider the irreps of the groups in the chain (15). Up to now, d was left unspecified. From now on, we shall assume that $2d \leqslant n$, or equivalently $d \leqslant \nu$†.

In the realisation (5), and (6), the $\mathrm{Sp}(2dn, R)$ group has only two (metaplectic) irreps. They are positive discrete series irreps, characterised by their lowest weight $\langle (1/2)^{dn}\rangle$ or $\langle (1/2)^{dn-1}3/2\rangle$, and their carrier space is the set of boson states with an even or odd boson number respectively. In the following, $\mathrm{Sp}(2dn, R)$ will play the role of H, and μ will be either one of its two irreps.

The irreps of $\mathrm{G}(n)$ are finite dimensional, and characterised by some partition $(\lambda_1\lambda_2\ldots\lambda_\nu)$‡. Those of $\mathrm{G^c}(2d)$ are positive discrete series, specified by their lowest weight $\langle \lambda'_d+\frac{1}{2}n, \ldots, \lambda'_1+\frac{1}{2}n\rangle$, where $(\lambda'_1\lambda'_2\ldots\lambda'_d)$ is also some partition. Their lowest weight state $P(\eta_{is})|0\rangle$ satisfies the following system of equations

$$
\begin{aligned}
&D_{ij}P(\eta_{is})|0\rangle = 0, && i\leqslant j(\varepsilon=+1) \text{ or } i<j(\varepsilon=-1),\\
&E_{ij}P(\eta_{is}|0\rangle = 0, && i>j,\\
&E_{ii}P(\eta_{is})|0\rangle = (\lambda'_{d+1-i}+\tfrac{1}{2}n)P(\eta_{is})|0\rangle. &&
\end{aligned}
\tag{18}
$$

From it the remaining states of the carrier space are obtained by applying the raising generators E_{ij}, $i<j$, and $D^\dagger_{ij}$, $i\leqslant j$ $(\varepsilon=+1)$ or $i<j$ $(\varepsilon=-1)$.

It was shown elsewhere with some specific choices for the metric g (Chacón 1969, Quesne 1973) that (18) admits a solution which is at the same time the highest weight state of some $\mathrm{G}(n)$ irrep $(\lambda_1\lambda_2\ldots\lambda_\nu)$ if and only if $\lambda_1=\lambda'_1, \ldots, \lambda_d=\lambda'_d$, $\lambda_{d+1}=\ldots=\lambda_\nu=0$, and that under these conditions the solution is unique. Since a similar proof can be devised whenever using an arbitrary symmetric or antisymmetric metric, we may state the following result: both $\mathrm{O}(n)$, $\mathrm{Sp}(2d, R)$, and $\mathrm{USp}(n)$, $\mathrm{SO}^*(2d)$ form a pair of complementary groups within either irrep $\langle(1/2)^{dn}\rangle$ or $\langle(1/2)^{dn-1}3/2\rangle$ of $\mathrm{Sp}(2dn, R)$.

As a final point, let us establish the existing relation between the first-order Casimir operators of $\mathrm{G}(n)$ and $\mathrm{G^c}(2d)$, the latter being respectively defined by

$$\Phi = -\tfrac{1}{2}\sum_{sts't'} g_{ss'}g_{tt'}\Lambda_{st}\Lambda_{s't'}, \tag{19}$$

and

$$\Phi^{\mathrm{c}} = \sum_{ij}[E_{ij}E_{ji}-\tfrac{1}{2}(D^\dagger_{ij}D_{ij}+D_{ij}D^\dagger_{ij})]. \tag{20}$$

The eigenvalues of Φ corresponding to an irrep $(\lambda_1\lambda_2\ldots\lambda_\nu)$ is given by

$$\varphi = \sum_{s=1}^{\nu}\lambda_s(\lambda_s+n-2s+1-\varepsilon), \tag{21}$$

while that of Φ^{c} associated with an irrep $\langle\lambda'_d+\frac{1}{2}n, \ldots, \lambda'_1+\frac{1}{2}n\rangle$ is

$$\varphi^{\mathrm{c}} = \sum_{i=1}^{d}(\lambda'_i+\tfrac{1}{2}n)(\lambda'_i+\tfrac{1}{2}n-2i+1-\varepsilon). \tag{22}$$

† d can be interpreted as the maximum row number of the $\mathrm{G}(n)$ irreps we are interested in. For a given row number, we could of course consider higher values of d, but this would unnecessarily complicate the discussion to follow.

‡ In the $\mathrm{O}(n)$ case, we also have the associate irreps which have more than ν rows.

By using (1), (2), (5), (6), and (9), both Φ and Φ^c can be written in terms of the boson creation and annihilation operators. By reordering the latter, it is then straightforward to prove that the difference of the two Casimir operators is a multiple of the unit operator:

$$\Phi - \Phi^c = \tfrac{1}{4}nd(2d - n + 2\varepsilon). \qquad (23)$$

Equation (23) can also be directly checked on the eigenvalues φ and φ^c once the relation between the complementary irreps is taken into account.

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